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APPROXIMATE DETERMINATION OF STRESSES AND DISPLACEMENTS NEAR A ROUNDED NOTCH

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Abstract—A numerical method is presented for the determination of the stress and displacement fields in two-dimensional bodies with high stress concentrations caused by the presence of a rounded notch. The problem is formulated in terms of boundary integral equations of a type used previously by Barone and Robinson to study the effects of sharp notches and cracks. Fundamental solutions are found numerically which resemble the Williams' solutions for sharp notches, but satisfy the boundary conditions on the curved notch boundary. The coefficients of these modified Williams' solutions are treated as additional generalized unknowns in the integral equation formulation. The corresponding singular solutions used are those developed by Barone and Robinson.

1. INTRODUCTION

A method is described for determination of stresses and displacements in two-dimensional elastic bodies with high stress concentrations. The formulation of the problems is based on a boundary integral equation (BIE) technique. Major emphasis will be placed on treating boundary value problems for bodies where at some points the radius of curvature of the boundary becomes very small. Example of such configurations are notches where the root of the notch is curved and parallel-sided cracks of finite but small width. In this paper the method will be described for the rounded notch problem. However, with some modifications the method is applicable to the problem of the crack of finite width as well.

The material of the body is assumed to be homogeneous, isotropic and linearly elastic. The displacements are taken small enough that the linear theory of elasticity is applicable.

In recent years the boundary integral equation technique has been applied to elasticity problems of this type (see, e.g. Refs. [6–10]). The solution of the problems under consideration using the ordinary BIE method requires the introduction of many points in a small, localized region of the boundary. If too many points are used near the singularity, the method can give results which appear to be unstable with respect to the location of these points. Good results can be obtained with comparatively few points, but if high accuracy is required an alternative to the straightforward addition of more and more points becomes desirable. Several methods for resolving this difficulty have been proposed [1, 11, 12].

The overall strategy of the proposed procedure is to modify the process which has been found to work for the sharp notch [1]. The modification will run into difficulty only if the radius of the notch is not small with respect to an overall dimension of the body, in which case the ordinary BIE method is economically applicable, and one would not speak of a stress-concentration problem.

The method is based on the Barone and Robinson solution technique for sharp notches [1]. In that method the displacement field in the vicinity of the singular point (the tip of the notch) is represented by a series consisting of a linear combination of Williams' solutions for the sharp notch [2–4]. The unknown coefficients weighting the individual Williams' solutions are considered to be generalized displacements. Auxiliary solutions are then developed each of which picks out a single generalized displacement by Betti's law and relates this generalized displacement to the far-field boundary values of the body under consideration. However, before Betti's law can be applied to the whole body, a piece of radius ϵ centered at the singular point must be removed from the body. Eventually the limit $\epsilon \to 0$ is taken.

In the Barone and Robinson solution technique the Williams' solutions are used to describe

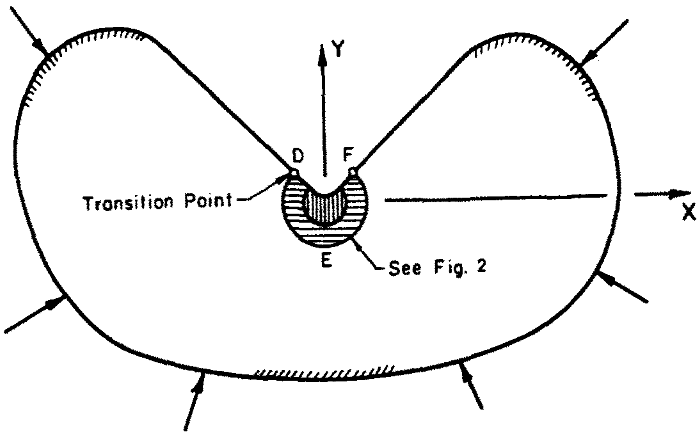

Fig. 1. A body with a notch having a fillet of small radius.

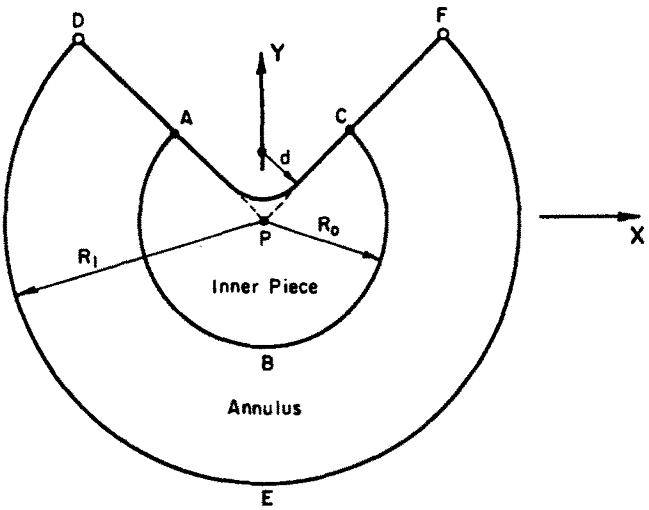

Fig. 2. Detail of the fillet region.

the field from the singular point out to the so-called transition points. Beyond the transition points, the usual "Kelvin" kernel of the ordinary BIE method is used.

In the proposed method for the rounded notch, three regions of the body are considered:

i. A small inner piece containing the fillet and bounded by a circle centered at the intersection of the two straight edges of the notch (see Figs. 1 and 2);

ii. An annular region adjacent to the inner piece as shown in Figs. 1 and 2; and

iii. The outer body, or the larger body, which consists of the rest of the body exterior to the annulus and the annulus itself.

The inner piece is treated numerically. For the annular region an analytical solution is developed using Williams' solutions which must be extended to include the so-called improper Williams' solutions.† Finally the larger body is analyzed using the Barone and Robinson solution technique where the generalized displacements express the field in the annular region. These coefficients tie the annulus to the rest of the body.

Here, in contrast to the Barone and Robinson solution technique, a finite inner piece must be used. The numerical solution in this inner piece reflects the details of the fillet.

2. THE MODIFIED WILLIAMS' SOLUTIONS

In the problem of the sharp notch, the Barone and Robinson Method guarantees that the singularity is indeed a sharp notch by the presence of only the proper Williams' solutions multiplying the generalized displacements in the region near the notch. In the case of the fillet, any selection of generalized displacements for the annular region must similarly correspond to

†These "improper" Williams' solutions correspond to unbounded strain energy in the region near the singular point.

the presence of the actual fillet boundary. In order to guarantee that this is the case, a set of numerical solutions is performed for the inner body, using the ordinary BIE Method.

A complete set of tractions is chosen for the boundary (*ABC*, Fig. 2) of the inner piece. For each member of the set, the corresponding displacements on *ABC*, (Fig. 2) are calculated along with any quantity eventually desired in the inner piece. This numerically determined field for the inner piece is called a "unit field." The displacements and tractions on the boundary *ABC* permit complete determination of the field in the annulus which matches the unit field of the inner piece on their common boundary.† The number of unit fields used in the actual numerical analysis must of course, be finite. Numerical experience in Ref. [2] indicates that 10–12 unit fields corresponding to the tractions of the proper Wiliiams' solutions gives excellent results for the expansion of the traction on the outer boundary of the inner body.

The solution in the annular region corresponding to a unit field is termed a "modified Williams' solution." Each modified Williams' solution is a linear combination of proper and improper Williams' solutions, i.e. a series of the form:

$$\begin{aligned}
D_{\rho m} &= \sum_{j=-N}^{M} \alpha_{mj} u_{\rho j}; \quad u_{\rho j} = A_j(\phi)\rho^{\lambda_j} \\
D_{\phi m} &= \sum_{j=-N}^{M} \alpha_{mj} u_{\phi j}; \quad u_{\phi j} = B_j(\phi)\rho^{\lambda_j} \\
S_{\rho\rho m} &= \sum_{j=-N}^{M} \alpha_{mj} \sigma_{\rho\rho j}; \quad \sigma_{\rho\rho j} = P_j(\phi)\rho^{\lambda_j-1} \\
S_{\phi\phi m} &= \sum_{j=-N}^{M} \alpha_{mj} \sigma_{\phi\phi j}; \quad \sigma_{\phi\phi j} = Q_j(\phi)\rho^{\lambda_j-1} \\
S_{\rho\phi m} &= \sum_{j=-N}^{M} \alpha_{mj} \sigma_{\rho\phi j}; \quad \sigma_{\rho\phi j} = R_j(\phi)\rho^{\lambda_j-1}, \quad j \neq 0,1,2
\end{aligned} \tag{1}$$

where the α_{mj}'s are the coefficients to be found from the *m*th unit field and the functions are the eigenfunctions corresponding to the sharp notch (see Appendix A). They comprise real as well as complex eigenfunctions. The scripts $j<0$ are used to represent the eigenfunctions with the eigenvalues $\lambda_j<0$; similarly $j\geq 0$ means $\lambda_j \geq 0$. The specification $j\neq 0,1,2$ excludes rigid-body motions.

Betti's law is used for the determination of the α_{mj}'s, the unknown coefficients of the *m*th modified Williams' solution. The objective is to connect the fields of the inner and annular regions smoothly.

Consider a smaller annulus in the body with outer radius $R_2 < R_1$ ($ABCA'B'C'$, Fig. 3). This piece is loaded along the boundary *ABC* with the tractions of the *m*th unit field of the inner piece, and along the boundary $A'B'C'$ with the tractions of the series representation of the same solution. The *k*th improper Williams' solution is now chosen as an auxiliary.

Betti's law applied to this annular piece results in

$$\Gamma_{mk} = \sum_{j=-N}^{M} \alpha_{mj} F_{jk} R_2^{\lambda_j+\lambda_k^*}, \tag{2}$$

$$\Gamma_{mk} = \int_{\phi=0}^{\phi=\phi_0} (\{D_{\rho m}\sigma^*_{\rho\rho k} + D_{\phi m}\sigma^*_{\rho\phi k}] - [S_{\rho\rho m}u^*_{\rho k} + S_{\rho\phi m}u^*_{\phi k}]\}R_0 \, d\phi \tag{3}$$

where R_0 is the radius the inner piece and the F_{mk} is given by eqn (B5) in Appendix B. It is also shown in that Appendix that:

$$F_{jk} = 0 \quad \text{for} \quad j \neq k \quad \text{or} \quad \lambda_k^* + \lambda_j \neq 0 \tag{4}$$

$$F_{jk} \neq 0 \quad \text{for} \quad j = k \quad \text{or} \quad \lambda_k^* + \lambda_j = 0. \tag{5}$$

Therefore,

$$\alpha_{mj} = \Gamma_{mj}/F_{mjj} \quad -N \leq j \leq M, \quad j \neq 0,1,2. \tag{6}$$

†The determination of the field from data on just one boundary of the annulus causes no difficulty (see Ref. [2]).

As can be seen, the α_{mj}'s are functions of the outer radius R_0, of the inner piece only. A question naturally arises: What should be the size of the inner piece? As mentioned earlier, the inner piece must be large enough to include the whole fillet so that the boundaries in the annular region are straight. This will insure homogeneous boundary conditions for the edges of the annular region. The inner piece should also be small enough that the stress gradients are not much larger than the maximum stress divided by the diameter. This should guarantee accurate results by the ordinary numerical integral equation process. The first condition leads to

$$R_0 \geq \frac{r}{\tan \alpha} \tag{7}$$

where r is the radius of the fillet and α is the half-opening angle of the notch. It has been found useful in Ref. [2] to take R_0 to be 2.5 times this minimum value for an opening angle of 90°.

Once the unknown coefficients of the modified Williams' solution are determined, we have a solution for the annular region which takes account of the fillet of the inner piece. This solution can be written in series form

$$u_\rho = \sum_{m=1}^{M} K_m D_{\rho m} \qquad u_\phi = \sum_{m=1}^{M} K_m D_{\phi m}$$

$$\sigma_{\rho\rho} = \sum_{m=1}^{M} K_m S_{\rho\rho m} \qquad \sigma_{\rho\phi} = \sum_{m=1}^{M} K_m S_{\rho\phi m}$$

$$\sigma_{\rho\phi} = \sum_{m=1}^{M} K_m S_{\rho\phi m} \tag{8}$$

where the K's are generalized displacements and the expansion functions are the modified Williams' solution. Here M, the total number of modified functions for the solution of a specific problem, must be taken equal to the number of proper Williams' solutions in each modified Williams' solution. Note that the first three functions in the series are the displacement field for rigid body motions which are identical for both modified and original Williams' solutions.

3. SOLUTION FOR THE LA2RGE BODY (LOADED BODY)

The solution procedure for the large body is essentially the same as was described in Appendix A for the solution of the body with a sharp notch using the BIE method. In fact, the same improper Williams' solutions used as auxiliaries for the sharp notch problem also serve as auxiliaries for the large body formed by the removal of the inner piece.† These auxiliaries will pick out the unknown generalized displacements given in eqn (8).

For evaluation of integrals along the boundary formed by the removal of the finite inner piece (Fig. 6), an equivalent path of integration can be chosen in order to simplify the computation of the integrals. It is shown in Ref. [2] that one simple path of integration consists of the edges of the fictitious sharp notch formed by the extension of the two straight parts of the boundary of the inner piece (see Ref. [2]).

Since some of the solutions are unbounded at point P (Fig. 2) a small piece of radius ϵ centered at this point is removed from the body before application of the Betti's law to the large body. The limit is then taken as $\epsilon \to 0$.

Once the integral equations are approximated and final linear equations solved for the unknown generalized displacements K's and other unknown boundary values, the displacements and stresses at any point in the annulus can be determined using eqn (8). The unknown fields for points in the inner piece and the larger body can be determined by eqn (A8) using the Kelvin kernel with already computed boundary data.

Chief interest in the numerical results will, of course, center in the inner piece. If a stress quantity of interest has been found for each unit field, a simple superposition will give the final stresses. Alternatively, one can compute the actual boundary tractions and displacements for the inner piece and use an ordinary BIE quadrature to arrive at the answers.

†Here, just as the case of the sharp notch, the Kelvin auxiliaries are used to pick out the rigid body translations.

Results desired in the outer body are obtained using a body consisting of the annulus and the outer body. An ordinary Kelvin kernel, or some combination of derivatives of it, is used as an auxiliary.

In the annulus, the entire field is available in an analytical form as a weighted sum of modified Williams' solutions.

4. NUMERICAL IMPLEMENTATION OF THE PROPOSED METHOD OF THE BOUNDARY VALUE PROBLEMS WITH TRACTION PRESCRIBED

We have seen that the solution of a given problem involves two types of integral equations. One of these types is written for the regular boundary points and the other type for the part of the boundary where the solution is represented by the extended Williams' solutions. The integral equations are then approximated by a set of linear algebraic equations. The system can be presented in matrix form

$$[A]\{u\} = \{B\} \tag{9}$$

where $[A]$ is the coefficient matrix. The entries of the column vector u are all the unknowns of the problem, the generalized displacements, and the unknown boundary displacements at regular points. The vector B is known.

Before solving eqn (9) for the unknowns, the rigid body displacements must be specified; otherwise, the matrix [A] is singular. This is done by setting three unknown displacements equal to zero and eliminating their corresponding equations from the eqn (9). These constraints must be such that they prevent rigid body motions without inhibiting any strains due to loading.

Before calculations can be carried out, a scale must be chosen for the eigenfunctions and

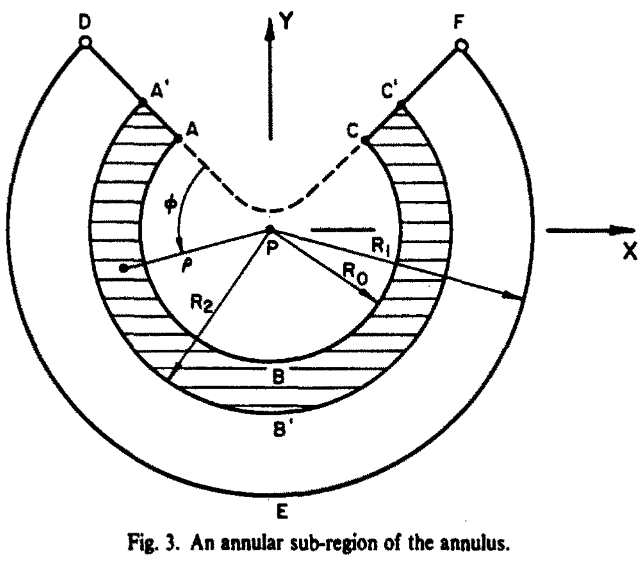

Fig. 3. An annular sub-region of the annulus.

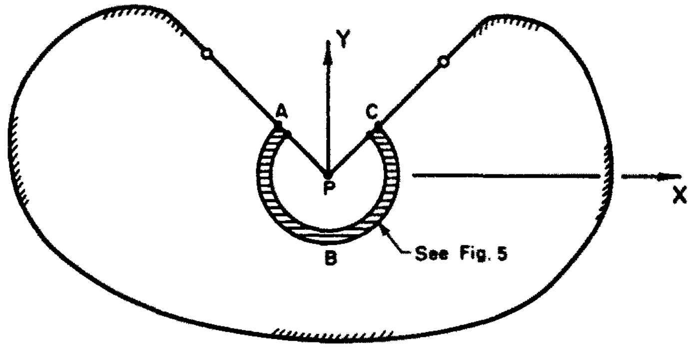

Fig. 4. An annular section of a body with a notch for development of the orthogonality condition.

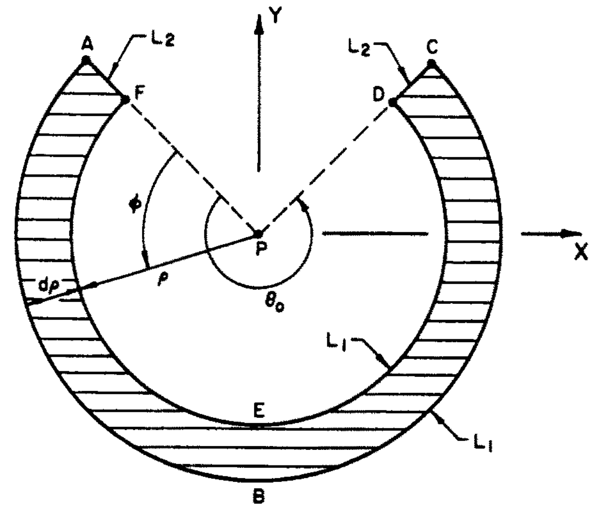

Fig. 5. Detail of the annular region for development of orthogonality conditions.

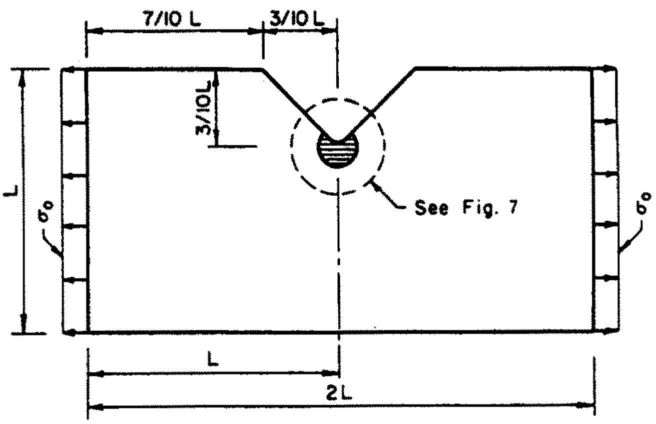

Fig. 6. A uniformly loaded plate with a notch having a fillet.

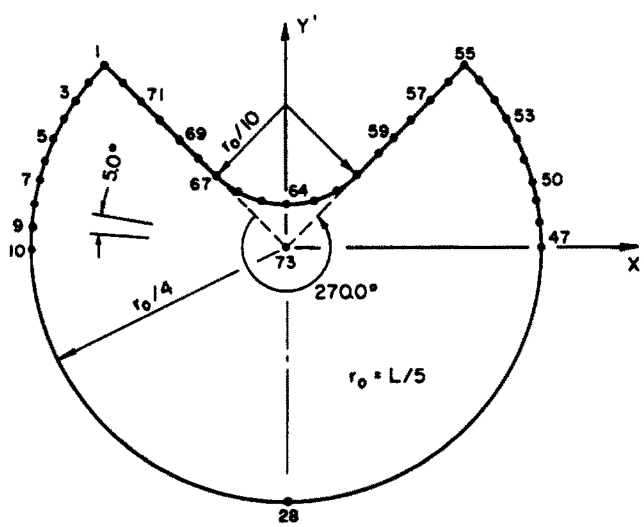

Fig. 7. Detail of the inner piece.

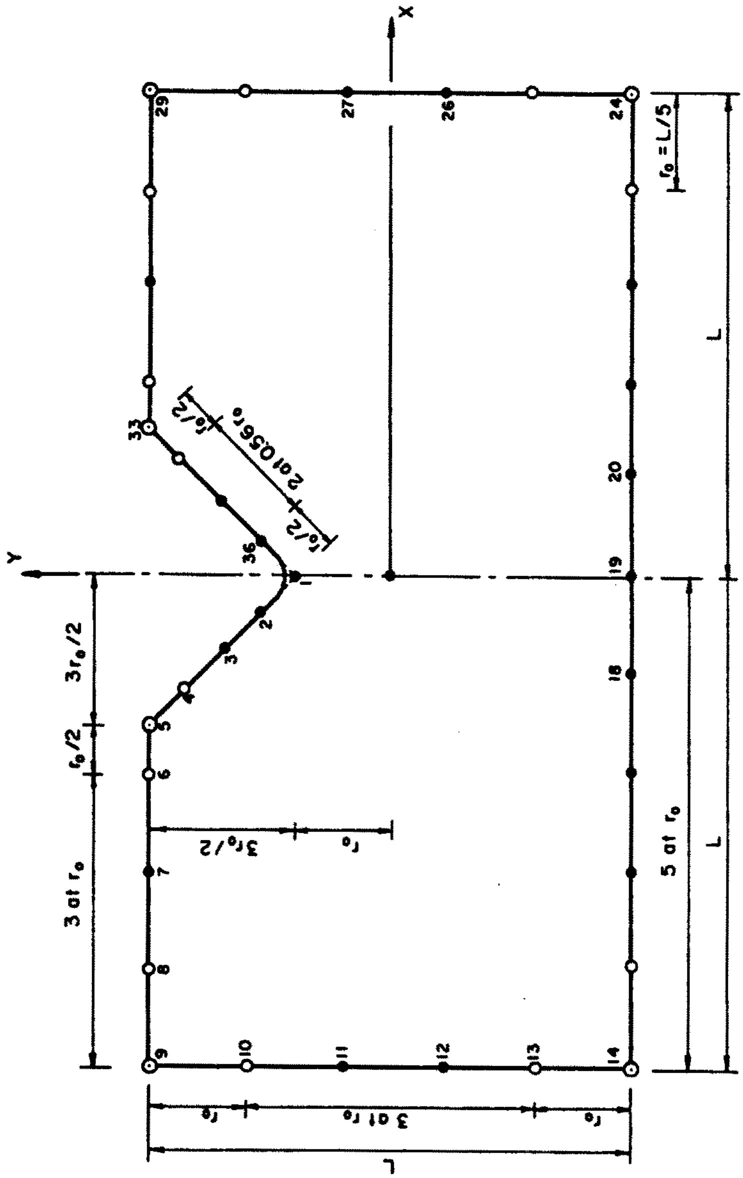

Fig. 8. Data for the sample problem.

Table 1. Expansion coefficients for certain modified Williams' solutions

Mode No.	Eigenvalue λ	Expansion Coefficients: Solution No. 4 $\lambda_0^2 = 0.54448$	Solution No. 5 $\lambda_0 = 1.62926+i0.23125$	Solution No. 6 $\lambda_0 = 1.62926+i0.23125$
4[1]	0.54448	0.769900	-0.000666	-0.000431
5	1.62925+i0.23125	-0.708000	0.978813	-0.002572
6		-0.002695	-0.001012	0.997131
7	2.97184+i0.37393	0.343755	0.007658	-0.007378
8		0.185256	0.006627	0.002075
9	4.31038+i0.45549	0.527339	0.020260	0.005180
10		-0.035010	0.001954	0.009042
11	5.64712+i0.54368	0.809780	0.041734	0.037121
12		-0.463231	0.014679	0.006847
13	-0.54448	-0.013316	-0.000374	0.000024
14	-1.62926-i0.23125	-0.000028	-0.000008	0.000004
15		0.000199	0.000082	0.000002

[1]The first three modes corresponding to rigid body motions are not included in the expansion.
[2]λ_0 is corresponding eigenvalue of the applied traction used to develop the unit field.

Table 2. Comparison of boundary data of selected unit fields and the modified Williams' solutions

Boundary Points[1]	Solution No. 4 $\lambda_0^2 = 0.54448$: Traction t_x/t_{x2}[3] Unit Field	Traction t_x/t_{x2}[3] Expansion	Displ. u_x/α[4] Unit Field	Displ. u_x/α[4] Expansion	Solution No. 5 $\lambda_0 = 1.62926+i0.23125$: Traction t_x/t_{x2} Unit Field	Traction t_x/t_{x2} Expansion	Displ. u_x/α Unit Field	Displ. u_x/α Expansion
2	1.0000	1.0082	0.63523	0.63548	1.0000	1.0054	2.64236	2.64218
4	1.2116	1.2103	0.59563	0.59612	2.9468	2.9355	1.70988	1.71063
6	1.3990	1.4033	0.55616	0.55640	4.6293	4.6330	0.79091	0.79091
8	1.5556	1.5524	0.52125	0.51845	5.9625	5.9800	0.06009	0.06084
10	1.6676	1.6710	0.48374	0.48322	6.8963	6.8888	0.79654	0.79616
12	1.7239	1.7350	0.44910	0.44893	7.4083	7.3890	1.38089	1.38052
14	1.7165	1.7179	0.41260	0.41252	7.5045	7.5075	1.78761	1.78799
18	1.4966	1.4940	0.32539	0.32576	6.5596	6.5583	2.02872	2.02834
24	0.7079	0.7071	0.14479	0.14489	3.0482	3.0520	1.11313	1.11313

[1]See Fig. 8
[2]λ_0 is the eigenvalue of the traction of the unit field.
[3]t_{x2} is traction at point 2, (See Fig. 8)
[4]$\alpha = \frac{t_{x2}}{E} r_0$ where E is Young modulus of elasticity and r_0 is shown in Fig. 9.

other extended Williams' solutions. In the problems under consideration, the magnitude of the vector displacement for each extended Williams' solution is set equal to r_0 at $\theta = 0$ and $\rho = r_0$ where r_0 is shown in Fig. 8.

In the problem solved the values are given in terms of r_0, the applied load σ_0 and the Young modulus e. Poisson's ratio is taken as $\nu = 0.25$.

The sample problem solved is shown in Figs. 6–8. The field in the annular region is represented by four real† and four pairs of complex modified Williams' solutions.

Table 1 gives the expansion coefficients for the first three modified Williams' solutions using

†Including two rigid body translation and one rotation.

Table 3. Generalized displacements

Mode No.	Eignevalue λ^1	(Generalized Displacements)/α^2
1	0.00	0.000
2	0.00	6.719
3	1.00	0.000
4	0.54448	6.140
5	1.6293+i0.2313	4.767
6		-0.274
7	2.9718+i0.3739	-2.169
8		-1.169
9	4.3104+i0.4555	-0.338
10		-0.029
11	5.64712+i0.5137	-0.535
12		0.326

[1] λ corresponds to the applied traction of the unit field.

[2] $\alpha = \frac{\sigma_0}{E}$ where σ_0 is the applied load and E is the modulus of elasticity.

Table 4. Selected boundary displacements

Point No.[2]	(Displacements)/α^1			
	The Proposed Method		Ordinary BIE Method	
	U_x	U_y	U_x	U_y
1	0.000	6.331	0.000	6.330
2	-3.015	5.664	-3.018	5.635
3	-4.658	4.897	-4.656	4.887
4	-5.827	4.153	-5.829	4.149
5	-6.578	3.516	-6.581	3.510
9	-8.996	-0.699	-8.999	-0.700
11	-6.533	0.000	-6.533	0.000
13	-4.214	0.778	-4.214	0.776
17	-0.593	4.726	-0.592	4.718
19	0.000	6.025	0.000	6.017

[1] $\alpha = \frac{\sigma_0}{E} r_0$ where σ_0 is the applied load, E is Young modulus of elasticity and r_0 is shown in Fig. 8.

[2] See Fig. 8.

nine proper and three improper eigenfunctions. Table 2 gives one component of displacements and tractions of the first two unit fields along with their corresponding values from the modified Williams' solutions. It can be seen from the table that the agreement is excellent. Table 3 gives the generalized displacements of the modified Williams' solutions for the annular region (see Fig. 4).

The displacements for some boundary points are given in Table 4. The same table gives the results from the solution of the fillet problem using the ordinary BIE method with 50 boundary points. The stresses and displacements for certain points of the inner piece are given in Table 5. Finally, the stress distribution along the axis of symmetry is given in Fig. 9.

Table 5. Stresses and displacements at selected points of the inner piece

Point[1] No.	Values from: 1. The Proposed Method 2. Ordinary BIE Method				
	Stresses			Displacements	
	σ_{xx}/σ_0[2]	σ_{yy}/σ_0	σ_{xy}/σ_0	U_x/α[3]	U_y/α
1	1.363	1.363	1.363	-1.962	6.002
	1.372	1.374	1.363	-1.960	5.998
10	3.464	0.520	-0.764	-1.123	5.823
	3.459	0.526	-0.772	-1.120	5.820
28	2.842	1.632	0.000	0.000	6.181
	2.842	1.628	0.000	0.000	6.180
64	13.522	0.002	0.000	0.000	6.265
	13.425	0.144	0.000	0.000	6.259
73	7.430	2.219	0.000	0.000	6.331
	7.408	2.206	0.000	0.000	6.330

[1] See Fig. 7.

[2] σ_0 is the applied load.

[3] $\alpha = \frac{\sigma_0}{E} r_0$ where E is modulus of elasticity and r_0 is shown in Fig. 8.

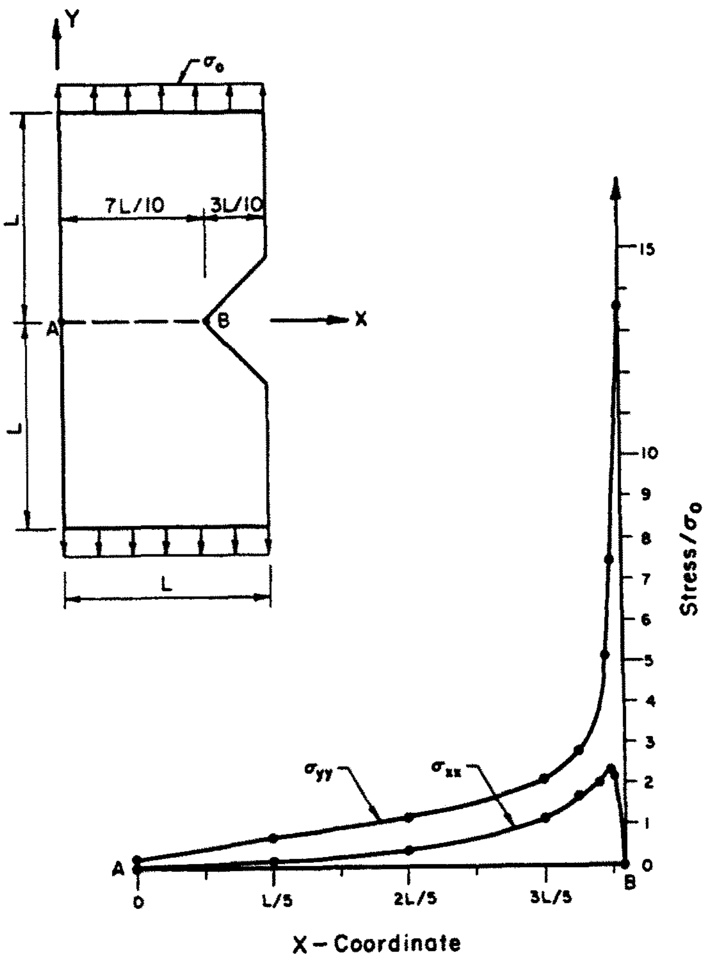

Fig. 9. Computed stresses for section AB.

The solution of the problem was found to be stable with respect to the number and location of boundary points and the number of extended Williams' solutions used in the expansion expressions. This is also the general conclusion reached by Barone[1] for the infinitesimal crack and sharp notch problems. This stability reflects the fact that the proposed integral equation approach leads to a strongly diagonal system of linear algebraic equations. The accuracy of the integral approximation is also an important factor. Certain numerical refinements such as subdividing the boundary intervals and including the boundary curvature in the evaluation of the singular integrals played a particularly important part in achieving accurate results.

The ordinary BIE method, by contrast, gave results that depended on the number of boundary points chosen and their location. As mentioned before this numerical difficulty is not unexpected as the notch sharpens. Certainly, if a moderate number of equations is to be used with usual precision of computation, the ordinary BIE method fails for this technically important type of problem. The proposed method involves no difficulties as the notch sharpens. It might be mentioned that the unit fields can be calculated once and for all for any single notch opening angle, that is, for all geometrically similar notches.

6. CONCLUSIONS

The general conclusion from the notch stresses found in Ref. [2] is that the method developed gives accurate and numerically stable results with rather small computational effort. The solution of the problems is fairly insensitive to the number of extended Williams' solutions used for the region with high concentration of stresses. The solution also is not affected very much be the number of boundary points used. Instead, it depends more on the method of approximation used for the evaluation of the integrals. In particular, a larger interval length can be used provided there is finer subdivision for the evaluation of the integrals. However, no quantitative analysis has been carried out to determine the optimum interval length and minimum computing time.

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REFERENCES

1. M. Barone and A. R. Robinson, Approximate determination of stresses near notches and corners in elastic media by an integral equation method. *Int. J. Solids Structures* **8**, 1319–1338 (1972).
2. H. Nikooyeh and A. R. Robinson, Boundary integral equation solution of plane elasticity problems with high stress concentrations, Civil Engineering Studies, *SRS* 461, University of Illinois at Urbana-Champaign, Urbana, Ill. (April 1979).
3. M. L. Williams, Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech* **19**, 526–528 (Dec. 1952).
4. M. L. Williams, The complex-variable approach to stress singularities. *J. Appl. Mech.* **23**, 477–478 (Sept. 1956).
5. O. K. Aksentian, Singularities of the stress-strain state of a plate in the neighborhood of an edge. *PMM, Applied Mathematics and Mechanics* 31(1), 193–202 (Aug. 1967).
6. F. J. Rizzo, An integral equation approach to boundary value problems of classical elastostatics. *Quartl. Appl. Math.* **25**, 83–95 (April 1967).
7. D. J. Forbes, Numerical analysis of elastic plates and shallow shells by an integral equation method. Civil Engineering Studies, *SRS* 345, University of Illinois at Urbana-Champaign, Urbana, Ill. (July 1969).
8. T. A. Cruse, Numerical solutions in three dimensional elastostatics. *Int. J. Solids Structures* **5**, 1259–1274 (1969).
9. T. A. Cruse, Boundary-integral equation fracture mechanics analysis, boundary-integral equation method. *Computational Applications in Applied Mechanics.* AMD-Vol. 11. (Edited by T. A. Cruse and F. J. Rizzo), American Society of Mechanical Engineers (1975).
10. T. A. Cruse, Numerical evaluation of elastic stress intensity factors by the boundary integral equation method. In *The Surface Crack: Physical Problems and Computational Solutions*, American Society of Mechanical Engineers, New York (1972).
11. M. Stern and M. L. Soni, On the computation of stress intensities at fixed-free corners. *Int. J. Solids Structures* **12**(4), 331–337 (1976).
12. M. D. Snyder and T. A. Cruse, Boundary-integral analysis of cracked anisotropic plates. *Int. J. Fracture* **11**, 315–328 (1975).

APPENDIX A. THE WILLIAMS' SOLUTIONS

In the development of the proposed method, extensive use is made of the displacement field in the neighborhood of an irregular boundary point. Analytical results for determination of the character of the displacement and stress fields are well known[3,4]. A more recent approach by Aksentian[5], which is based on a three-dimensional elastic analysis of the problem, provides a somewhat more general solution of the problem in the form of asymptotic expressions. A brief description of the Williams' solution for two-dimensional bodies in plane strain will be given here.

The displacement field in the neighborhood of an irregular point in polar coordinates may be written in the form

$$u_\rho = A(\phi)\rho^\lambda$$
$$u_\phi = B(\phi)\rho^\lambda. \tag{A1}$$

Solving the differential equations of equilibrium for the functions $A(\phi)$ and $B(\phi)$ gives

$$A(\phi) = (\lambda - 3 + 4\nu)[c_1 \cos(\lambda - 1)\phi + c_2 \sin(\lambda - 1)\phi] + c_3 \cos(\lambda + 1)\phi + c_4 \sin(\lambda + 1)\phi$$
$$B(\phi) = (\lambda + 3 - 4\nu)[-c_1 \sin(\lambda - 1)\phi + c_2 \cos(\lambda - 1)\phi] - c_3 \sin(\lambda + 1)\phi + c_4 \cos(\lambda + 1)\phi \tag{A2}$$

where the c's are arbitrary constants of integration and ν is Poisson's ratio. The corresponding stresses in polar coordinates are:

$$\sigma_{\rho\rho} = P(\phi)\rho^{\lambda-1}$$
$$\sigma_{\phi\phi} = Q(\phi)\rho^{\lambda-1}$$
$$\sigma_{\rho\phi} = R(\phi)\rho^{\lambda-1} \tag{A3}$$

where

$$P(\phi) = 2\mu\lambda\{(\lambda - 1)[c_1 \cos(\lambda - 1)\phi + c_2 \sin(\lambda - 1)\phi] + c_3 \cos(\lambda + 1)\phi + c_4 \sin(\lambda + 1)\phi\}$$
$$Q(\phi) = -2\mu\lambda\{(\lambda + 1)[c_1 \cos(\lambda - 1)\phi + c_2 \sin(\lambda - 1)\phi] + c_3 \cos(\lambda + 1)\phi + c_4 \sin(\lambda + 1)\phi\}$$
$$R(\phi) = 2\mu\lambda\{(\lambda - 1)[-c_1 \sin(\lambda - 1)\phi + c_2 \cos(\lambda - 1)\phi] - c_3 \sin(\lambda + 1)\phi + c_4 \cos(\lambda + 1)\phi\} \tag{A4}$$

and μ is the shear modulus.

The constants c_i must be determined so that displacements and stresses satisfy the imposed boundary conditions. If the body is a wedge, boundary conditions will be given along the sides of the wedge. For example, specifications of two homogeneous tractions boundary conditions along each edge leads to an eigenvalue problem[3,4] with characteristic equation of the form

$$\sin^2 \lambda\theta_0 = \lambda^2 \sin^2 \theta_0 \tag{A5}$$

where θ_0 is the internal angle of the wedge. There are an infinite number of eigenvalues (λ_m, $m = 1, 2, 3, \ldots$) which satisfy eqn (A5) and lead to an infinite number of eigenfunctions, each of which is determined to within a multiplicative constant.

It was shown that there are an infinite number of displacement fields each of which satisfies the homogeneous boundary conditions. In general, the results occur in complex conjugate pairs of solutions, each pair leading to two real solutions. Detailed representation of these fields are given by Barone[1]. For simplicity of the arguments, the eigenfunctions will be considered in the form $\rho^\lambda\psi(\phi)$ even if λ is complex.

Any combination of the Williams' solutions is also a solution. That is, the following expressions are possible solutions of the homogeneous problem.

$$u_\rho = \sum_m K_m u_{\rho m}; \quad u_{\rho m} = A_m(\phi)\rho^{\lambda_m}$$
$$u_\phi = \sum_m K_m u_{\phi m}; \quad u_{\phi m} = B_m(\phi)\rho^{\lambda_m}$$
$$\sigma_{\rho\rho} = \sum_m K_m \sigma_{\rho\rho m}; \quad \sigma_{\rho\rho m} = P_m(\phi)\rho^{\lambda_m - 1}$$
$$\sigma_{\phi\phi} = \sum_m K_m \sigma_{\phi\phi m}; \quad \sigma_{\phi\phi m} = Q_m(\phi)\rho^{\lambda_m - 1}$$
$$\sigma_{\rho\phi} = \sum_m K_m \sigma_{\rho\phi m}; \quad \sigma_{\rho\phi m} = R_m(\phi)_\rho^{l_m - 1}. \tag{A6}$$

In the Barone and Robinson solution techniques[1], the K's serve as generalized displacements. In order to insure that the displacements in eqn (A6) are bounded, the real part of each eigenvalue λ_m must be greater or equal to zero. Such solutions will be termed "proper" Williams' solutions. However, in eqn (A5) if λ_m is a solution then $-\lambda_m$ is also a solution[1]. A solution with the real part of the eigenvalue less than zero will be called an "improper" Williams' solution. As will be seen, these improper solutions are used as auxiliaries for the determination of the generalized displacements. They may be expressed as

$$u_\rho^* = \sum_m K_m^* u_{\rho m}^*; \quad u_{\rho m}^* = A_m^*(\phi)\rho^{\lambda_m^*}$$
$$u_\phi^* = \sum_m K_m^* u_{\phi m}^*; \quad u_{\phi m}^* = B_m^*(\phi)\rho^{\lambda_m^*}$$
$$\sigma_{\rho\rho}^* = \sum_m K_m^* \sigma_{\phi\phi m}^*; \quad \sigma_{\rho\rho m}^* = P_m^*(\phi)\rho^{\lambda_m^* - 1}$$

$$\sigma^*_{\phi\phi} = \sum_m K^*_m \sigma^*_{\phi\phi m};\ \sigma^*_{\phi\phi m} = Q^*_m(\phi)\rho^{\lambda^*_m - 1}$$

$$\sigma^*_{\rho\phi} = \sum_m K^*_m \sigma^*_{\rho\phi m};\ \sigma^*_{\rho\phi m} = R^*_m(\phi)\rho^{\lambda^*_m - 1} \quad \text{(A7)}$$

It is shown in Ref. [1] that $\lambda^*_m = -\lambda_m$ where λ_m is a positive eigenvalue for eqn (A5).

The solution technique of Barone and Robinson

The formulation of the boundary integral equations for elastic bodies with irregular boundary points is given by Barone and Robinson[1]. In this method the proper Williams' solution eqn (A6), with K's as unknown generalized displacements are used to represent the displacement field in the vicinity of the irregular boundary point up to so-called transition points. The integral equation for the regular boundary points are then written in usual manner (see, e.g. Refs. [6–10]). For irregular boundary points a new kernel was derived which turns out to be the eigenfunctions of the improper Williams' solution eqn (A7). These auxiliary solutions are used to pick out the unknown generalized displacements. However, for the first two K's corresponding to rigid-body displacements the Kelvin auxiliary solutions are used.

Since the improper Williams' solutions are singular at an irregular point, a small circular piece of radius ϵ containing the irregular point is removed from the body before the reciprocal theorem is applied to the actual and auxiliary solution. The reciprocity relation for the body after taking the limit $\epsilon \to 0$ is

$$\sum_{m=3}^{M} -\alpha_{im}K_m + \int_L t^*_{ij}(P,Q)u_j(Q)\,ds(Q) = \int_L u^*_{ij}(P,Q)t_j(Q)\,ds(Q) \quad \text{(A8)}$$

where in eqn (A8) the standard summation convention is used on the subscript j. The $t^*_{ij}(P,Q)$ and $u^*_{ij}(P,Q)$ are tractions and displacements of the ith improper Williams' solution. The $t_j(Q)$ and $u_j(Q)$ are the actual traction and the displacement of the boundary point Q in the jth coordinate direction and P is the notch root at which the auxiliary solutions are singular. Here M is the total number of proper Williams' solutions used, including the three rigid-body motions.† Furthermore,

$$\alpha_{im} = 0 \quad \text{for} \quad i \neq m$$

$$\alpha_{im} = F_{im} \quad \text{for} \quad i = m, m \geq 3$$

where F_{im} is given by eqn (B5).

This means that the mth auxiliary solution picks out the mth generalized displacement and relates it to the far field boundary values. The ordinary integral equations at regular points along with these new ones form a coupled system involving the generalized displacements and other boundary values as unknowns. This leads to a strongly diagonal system of linear algebraic equations which is solved for the unknown values. As mentioned before, the generalized displacements depend on the boundary values at fairly remote points. This appears to contribute to a numerically stable solution. Physically, the generalized displacements must depend on the applied loads in the far field. The fact that this dependence is explicit in the numerical procedure is one advantage of the method.

APPENDIX B. ORTHOGONALITY CONDITIONS FOR THE WILLIAMS' EXTENDED SOLUTIONS

Consider a body with a sharp notch of interior angle θ_0 (see Figs. 4, 5). We remove the annulus $ABCDEF$ from the body as shown. Let us write Betti's law for the piece using two different Williams' solution forms.

$$W = W_{L_1} + W_{L_2} = 0; \quad \text{(B1)}$$

$$W_{L_i} = \int_{L_i} [t^*_j u_j - u^*_j t_j]\,ds \quad i = 1,2; \quad j = 1,2. \quad \text{(B2)}$$

In eqn (B2) the summation convention is used for the lower-case indices. The L_1 is the curved part of the boundary and L_2 consists of the two straight edges shown in Fig. 5.

Substituting for the tractions and displacement in eqn (B2) from eqn (A6), we get

$$W_{L_1} = F_{mn}[(\rho + d\rho)^{\lambda_m + \lambda^*_n} - \rho^{\lambda_m + \lambda^*_n}] \quad \text{(B3)}$$

or

$$W_{L_1} = F_{mn}[(\lambda_m + \lambda^*_n)\rho^{\lambda_m + \lambda^*_n - 1}\,d\rho]; \quad \text{(B4)}$$

$$F_{mn} = \int_{\phi=0}^{\phi=\theta_0} \{[A_m(\phi)P^*_n(\phi) + B_m(\phi)R^*_n(\phi)] - [A^*_n(\phi)P_m(\phi) + B^*_n(\phi)R_n(\phi)]\}\,d\phi$$

and,

$$W_{L_2} = G_{mn}\rho^{(\lambda_m + \lambda^*_n - 1)}\,d\rho; \quad \text{(B6)}$$

$$G_{nm} = H_{mn} \quad (\phi = \theta_0) - H_{mn} \quad (\phi = 0). \quad \text{(B7)}$$

Here

$$H(\phi) = [A_m(\phi)R^*_n(\phi) + B_m(\phi)Q^*_n(\phi)] - [A^*_n(\phi)R_m(\phi) + B^*_n(\phi)Q_m(\phi)] \quad \text{(B8)}$$

Substituting for W_{L_1} and W_{L_2} in eqn (B1) we obtain

$$[(\lambda_m + \lambda^*_n)F_{mn} + G_{mn}]\rho^{(\lambda_m + \lambda^*_n - 1)}\,d\rho = 0 \quad \text{(B9)}$$

†The two translations are "picked out" by Kelvin solutions. The rigid-body rotation is in the set of Williams' solutions.

Since eqn (B9) must be true for all values of ρ we have

$$(\lambda_m + \lambda_n^*)F_{mn} + G_{mn} = 0 \tag{B10}$$

We consider the two possible cases:

(a) $$G_{mn} = 0 \quad \text{then} \quad F_{mn} = 0 \quad \text{if} \quad \lambda_m + \lambda_n^* \neq 0 \tag{B11}$$

and for $\lambda_m + \lambda_n^* = 0$, eqn (B10) does not give any information about F_{mn}. Conversely, of the first term in eqn (B10) is zero, then $G_{mn} = 0$.

(b) $$G_{mn} \neq 0 \quad \text{then} \quad F_{mn} = -\frac{G_{mn}}{\lambda_m + \lambda_n^*} \quad \text{for} \quad \lambda_m + \lambda_n^* \neq 0 \tag{B12}$$

which shows that the integrals around the curved boundary can be evaluated by finding the work terms at the two end points and divide it by $\lambda_m + \lambda_n^*$.